Nonconstructive Tools for Proving Polynomial-Time Decidability

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Abstract. Recent advances in graph theory and graph algorithms dramatically alter the traditional view of concrete complexity theory, in which a decision problem is generally shown to be in P by producing an efficient algorithm to solve an optimization version of the problem. Nonconstructive tools are now available for classifying problems as decidable in polynomial time by guaranteeing only the *existence* of polynomial-time *decision* algorithms. In this paper these new methods are employed to prove membership in P for a number of problems whose complexities are not otherwise known. Powerful consequences of these techniques are pointed out and their utility is illustrated. A type of partially ordered set that supports this general approach is defined and explored.

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1. *Introduction*

Early results aimed at a proof of *Wagner's conjecture* [45] and a solution to the *k-disjoint connecting paths problem* [22] provided a novel and powerful tool [35, theorem 5.1] for guaranteeing membership in P. This tool has been used to prove the polynomial-time decidability of many problems, including some that are not directly about graphs [16]. Further progress [37] has produced polynomial-time algorithms for the disjoint-paths problem and the related problem of testing whether an input graph G contains a fixed graph H as a minor (more on this in Section 2).

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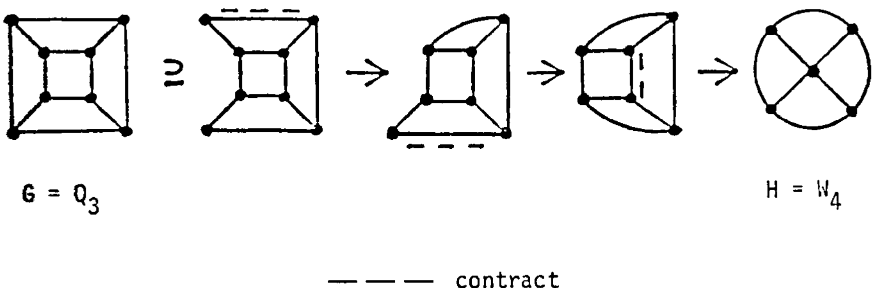


FIG. 1. Construction demonstrating that W_4 is a minor of Q_3.

An affirmative solution to Wagner's conjecture has recently been announced and proved [36, 38].

In this paper we explore how these developments constitute a new paradigm for proving polynomial-time decidability. In the next section we present the background information necessary to state these striking new advances, describe the resulting complexity tool, and discuss its use in guaranteeing membership in P. Section 3 comprises several applications of this method. We prove polynomial-time decidability for a number of combinatorial problems not otherwise known to be in P. In Section 4, we consider the importance of this fundamental tool in more general settings. The final section contains a brief discussion of some potential directions for this new line of research.

2. *Background*

All graphs we consider are finite and undirected, but may have loops and multiple edges. A graph H is called a *minor* of a graph G, written $H \leq G$, if a graph isomorphic to H can be obtained from a subgraph of G by contracting edges. For example, the construction depicted in Figure 1 shows that the graph of the wheel with four spokes is a minor of the graph of the three-dimensional binary cube.

Note that the relation $\leq$ defines a partial ordering on graphs. A family F of graphs is said to be *closed* under the minor ordering if the facts that G is in F and that H is a minor of G together imply that H must be in F. The *obstruction set* for a family F of graphs is defined to be the set of graphs in the complement of F that are minimal in the minor ordering. Therefore, if F is closed under the minor ordering, it has the following characterization: G is in F if and only if there exists no H in the obstruction set for F such that $H \leq G$.

THEOREM A [37]. *For every fixed graph H, the problem that takes as input a graph G and determines whether $H \leq G$ is solvable in polynomial time.*

THEOREM B [38] (Formerly known as Wagner's Conjecture). *Any set of finite graphs contains only a finite number of minor-minimal elements.*

Theorems A and B guarantee only the *existence* of a polynomial-time decision algorithm for any minor-closed family F of graphs. As of this writing, the proof of Theorem B is *nonconstructive*. That is, when we are able to apply Theorem B to (a finite description of) F, we are assured of a finite obstruction set for F without being given (by the arguments that establish the theorem) a means of identifying the elements of the set, the cardinality of the set, or even the order of the largest graph in the set.

There have previously been "nonconstructive" proofs of the existence of polynomial-time algorithms that turn out not to be inherently or mathematically nonconstructive. For example, it is argued in [14] that efficient algorithms exist by way of results from [24] for determining whether the achromatic number of a graph is at least k, for each fixed k, although these algorithms are exhibited only for a few small values of k. Nonetheless, it is easily observed that the algorithms *could* be produced by straightforward exhaustive computation. In contrast, it has been shown that *any* proof of Theorem B must be inherently nonconstructive in a precise, mathematically strong sense. See [21] for details.

Another interesting feature of Theorems A and B is the low degree of the polynomials bounding the decision algorithms' running times. Letting n denote the number of vertices in G, the general bound is $O(n^3)$. If F excludes even one planar graph, then the bound decreases to $O(n^2)$. (See [37] for details.) Curiously, as of this writing, these polynomials possess enormous constants of proportionality, rendering them impractical for problems of any nontrivial size [28]. Therefore, Theorems A and B can be viewed largely as results that constitute a tool for determining problem complexity, and thus serve to direct attention to a search for alternate, practical algorithms. Whether the algorithms directly promised by these theorems can be made effective in practice (assuming that the obstruction set for the family of graphs of interest is known) is an important open question.

3. *Applications of the Fundamental Tool*

We now employ Theorems A and B to prove the polynomial-time decidability of a number of illustrative problems not previously known to be in P (in some cases not even previously known to be decidable). We also demonstrate how the new tool provided by these theorems can be used to guarantee decision algorithms with low-degree polynomial running times for problems for which the best existing polynomial-time algorithms have high-degree polynomial running times.

One of the fundamental limitations of parallel computing is that the processors of a large network cannot all be directly connected. In such a computing environment, all that is logically required is that the necessary operands be locally available to every processor. For instance, suppose architectural requirements dictate that we perform parallel computing on some fixed surface s. We say that a graph H *emulates* a graph G if there is a surjective map $f\colon V(H) \to V(G)$ such that $f(x) = u$ and $uv \in E(G)$ implies that there is some y such that $f(y) = v$ and $xy \in E(H)$. For example, when s is the plane, the mapping depicted in Figure 2 illustrates that K_5 can be emulated by a planar graph. (For other applications of emulation, see [15].)

Hence we consider the following decision problem:

s-emulation

Instance: A graph G.

Question: Is there a graph H that embeds on s and emulates G?

THEOREM 1. *s-emulation can be decided in* $O(n^3)$ *time.*

PROOF. We shall show that the family F of "yes" instances is closed under the minor ordering. To do this, we must show that, if there is a graph H that embeds on s and emulates G, then for any G' that is a minor of G there is a graph H' that embeds on s and emulates G'.

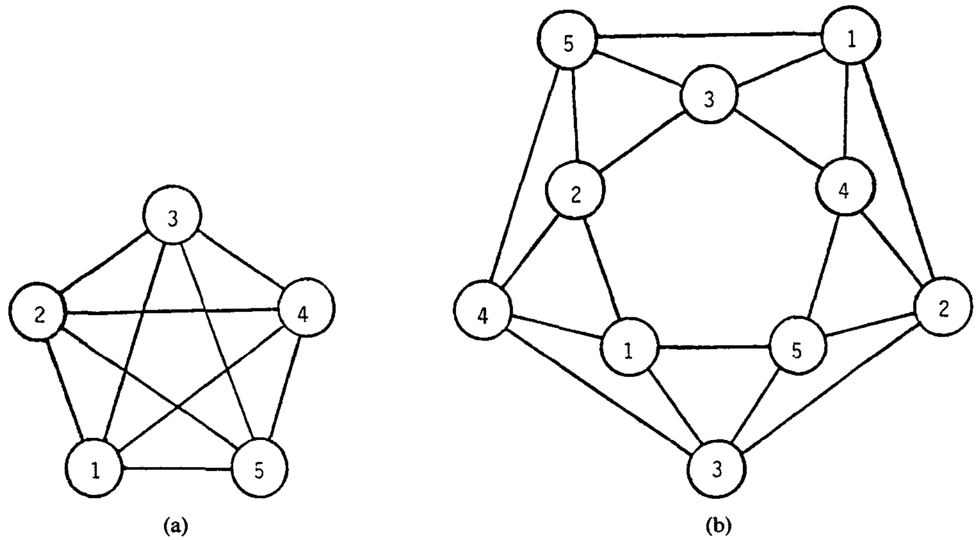


FIG. 2. Construction demonstrating that K_5 can be emulated in the plane. (a) $G = K_5$. (b) H.

Choose an embedding of H and an emulation map f: $V(H) \rightarrow V(G)$. It suffices to consider two cases. If G' is a subgraph of G, then the set of vertices $f^{-1}(V(G'))$ induces a subgraph H' of H that emulates G' by the map f' that restricts f to H', and clearly H' embeds on s. If G' is obtained from G by contracting an edge uv, then contracting every edge xy of H where $f(x) = u$ and $f(y) = v$ gives a graph H' that embeds on s and emulates G'. Therefore, in any case, G' is in F, and hence F is minor closed. □

It is important to note that even if we get a "yes" answer to *s-emulation* for a graph G, Theorem 1 does not guarantee that there is any effective method for finding a satisfactory graph H. We only know that at least one such H exists. Moreover, even if we are somehow given H and a suitable representation for the arbitrary fixed surface s, we are not assured of any efficient algorithm for finding an embedding of H on s.

The notion of covers of graphs (as 1-complexes, covering spaces [42]) has been used to model problems of distributed computing [2] and can be viewed as a restricted form of emulation. Combinatorially, we say that a graph H *covers* a graph G if there is a way to label the vertices of H with the vertices of G in such a way that, if vertex x of H is labeled u, then the neighbors of x in H are labeled in some one-to-one correspondence with the neighbors of u in G. For example, Figure 2 depicts a graph that covers as well as emulates K_5. Thus we have the following decision problem:

s-cover

Instance: A graph G.

Question: Is there a graph H that embeds on s and covers G?

THEOREM 2. *s-cover can be decided in $O(n^3)$ time.*

PROOF. Analogous to the proof of Theorem 1. □

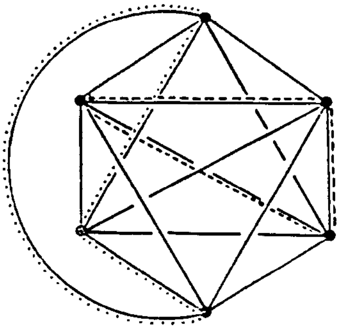
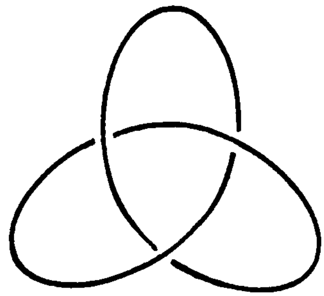


FIG. 3. Pair of topologically linked cycles in a 3-space embedding of K_6.

FIG. 4. A nontrivial knot (the trefoil).



The topological properties of graphs embedded in 3-space are a particularly rich source of problems amenable to this approach.

linklessness

Instance: A graph G.

Question: Can G be embedded in 3-space so that no two disjoint cycles of G are topologically linked (as in links of a chain, see Figure 3)?

This problem is nontrivial. For example, any planar graph can be embedded in 3-space without linked cycles, while it has been shown in [11] and [40] that every embedding of K_6 must have a pair of disjoint, topologically-linked cycles. With respect to decidability, there is a highly exponential algorithm that decides whether two cycles embedded in 3-space are linked [23, 41], and one could test all pairs of cycles in the graph, but it is not at all obvious how to account for all possible 3-space embeddings. Thus, it is not even clear how to show that *linklessness* is decidable without recourse to Theorems A and B.

THEOREM 3. *Linklessness can be decided in $O(n^3)$ time.*

PROOF. Clearly, a subgraph of a linklessly embedded graph is linklessly embedded. Contracting an edge of a linklessly embedded graph does not create links. Thus, "yes" instances are closed under minors. (Incidently, we have recently learned that Theorem 3 has been independently proved in [33a].) □

Similarly, we can show polynomial-time decidability for:

knotlessness

Instance: A graph G.

Question: Can G be embedded in 3-space so that all cycles of G are unknotted (see Figure 4)?

THEOREM 4. *Knotlessness can be decided in $O(n^3)$ time.*

PROOF. Analogous to the proof of Theorem 3. □

A problem that has recently attracted some attention is to determine the complexity, for fixed k, of the decision problem that asks whether an input graph G has a cycle [7] of length divisible by k (*(k)-cycles*; see [3] for a comprehensive survey). That this is decidable in polynomial time for every fixed k has recently been shown in [44]. Let g be a fixed, finite Abelian group and consider the following generalization of the complement of the (k)-cycle problem (i.e., is there no cycle in G whose length is divisible by k?).

g-unit cycle avoidance

Instance: A graph G.

Question: Can the edges of G be labeled by elements of g so that *no* cycle in G has the identity of g as the sum of the labels of its constituent edges?

In contrast to the *(k)-cycles* problem, neither *g-unit cycle avoidance* nor its complement belongs in any obvious way to NP. (For a concrete example, K_5 has the property that no matter how its edges are labeled with elements of the Abelian group of the integers modulo 3, there is a cycle whose label sum is 0.)

THEOREM 5. *For $g \simeq (Z_2)^k$, g-unit cycle avoidance can be decided in $O(n^3)$ time.*

PROOF. We shall show that "yes" instances are closed under minors. If G has a labeling that avoids unit cycles, then so does any subgraph of G. Suppose G' is obtained from G by contracting an edge uv to vertex u (eliminating v). Let E_u denote the neighbors of u in G exclusive of v. For a labeling l of the edges of G, we define a labeling l' for G' as follows:

$$l'(e) = \begin{cases} l(e) + l(uv) & \text{if } e \in E_u, \\ l(e) & \text{otherwise.} \end{cases}$$

We now prove that the sum of the labels for any cycle in G' also occurs as the sum of the labels for some cycle in G. Thus the set of graphs for which there is a labeling with no cycle summing to the identity is closed under minors.

If C' is a cycle in G', then corresponding to C' there is a cycle C in G that is taken to C' by the contraction of uv. (Incidentally, C is unique, because we allow loops and multiple edges.) If C contains the edge uv, then it contains exactly one edge from E_u; if C does not contain uv, then it contains either zero or two edges from E_u. In the first case the sum of the labels of C', $l'(C')$, is equal to $l(C) - l(uv) + l(uv) = l(C)$. In the second case $l'(C') = l(C)$ if C contains no edges from E_u, and $l'(C') = l(C) + 2l(uv) = l(C)$ if C contains two edges from E_u, because every element of a group isomorphic to $(Z_2)^k$ has order 2. □

We conclude this section with a generic form of fixed-parameter decision problem. Let F denote an arbitrary minor-closed family of graphs.

within k vertices of F

Instance: A graph G.

Question: Does G contain a set of k or fewer vertices that, when deleted, leave a graph in F?

For example, *k-vertex cover* and *k-feedback vertex set* are representatives of *within k vertices of F* when F is the family of edgeless graphs and the family of acyclic graphs, respectively.

For every fixed value of k and every minor-closed family F, this problem is trivially decidable in polynomial time by brute force. One need only check the $\binom{n}{k}$ graphs that result from the removal of k vertices. This can be done in $O(n^k p(n))$ time, where $p(n)$ bounds the time required to test for membership in F.

THEOREM 6. *Within k vertices of F can be decided in $O(n^3)$ time. Moreover, it can be decided in $O(n^2)$ time if the family of "yes" instances excludes a planar graph.*

PROOF. Let k denote any fixed positive integer and F any minor-closed family of graphs. Let $F(k)$ denote the family of "yes" instances for *within k vertices of F*. We shall argue that $F(k)$ is closed under minors as well, ensuring the time bounds in the statement of the theorem.

Suppose that G is a graph in $F(k)$ and, therefore, that there exists a set $S \subseteq V(G)$ with $|S| \leq k$, where $G - S$ is in F. If H is a subgraph of G, then $H - S$ is a subgraph of $G - S$; thus $H - S$ is in F, since F is closed under subgraphs. It follows that H is in $F(k)$.

It remains to show that $F(k)$ is closed under contractions. Suppose H is obtained from G by contracting the edge uv to the vertex u (eliminating v). Define

$$S' = \begin{cases} S & \text{if } S \cap \{u, v\} = \emptyset, \\ (S - \{v\}) \cup \{u\} & \text{otherwise.} \end{cases}$$

Then $|S'| \leq k$ and $H - S'$ is obtained from $G - S$ by contracting uv if $S \cap \{u, v\} = \emptyset$, while $H - S'$ is a subgraph of $G - S$ otherwise. In either case $H - S'$ is a minor of $G - S$ and is therefore in F. □

Since the family of edgeless graphs is obviously minor closed, and since a matching of size $k + 1$ is a planar "no" instance, it follows that *k-vertex cover* is decidable in $O(n^2)$ time. Similarly, since the family of acyclic graphs is obviously minor closed, and since $k + 1$ copies of K_3 is a planar "no" instance, it follows that *k-feedback vertex set* is decidable in $O(n^2)$ time. Results such as these provide motivation to search for direct, constructive quadratic-time algorithms. It has been pointed out by M. Blum and S. Rudich (personal communication) and an anonymous referee that k-vertex cover can in fact be solved directly in $O(n^2)$ time (by starting with a maximal matching). Additionally, we have found a direct method to decide k-feedback vertex set in $O(n^2)$ time (with the aid of an algorithm from [27] for finding an almost-shortest cycle in a graph).

A number of other problems yield to this same line of attack. The *k-search number* problem has been shown to be decidable in $O(n)$ time for trees and for $k \leq 3$ in [30], and in $O(n^{2k^2+4k+8})$ time for arbitrary fixed k in [13]. However, since the family of "yes" instances is easily seen to be closed under minors and there are trees with arbitrarily large search number, the problem is decidable in $O(n^2)$ time for arbitrary fixed k [17]. The same holds for important variations of this problem (C. H. Papadimitriou, personal communication). The *k-tree embedding* problem has been shown to be decidable in $O(n^{k+2})$ time in [4]. Nevertheless, its family of "yes" instances are clearly minor closed and exclude the $(k + 1) \times (k + 1)$ grid, ensuring the existence of an $O(n^2)$ time decision algorithm. The *disk dimension* problem [19] has been shown to be decidable in $O(n)$ time for genus $g = 0$ (the plane) and any fixed number of disks d in [6]. It has also been shown to be decidable in $O(n^{O(g)})$ time for fixed g and unbounded d in [20]. Again, however, the family of "yes" instances is minor closed and excludes a sufficiently large grid, thereby giving rise to an $O(n^2)$ time decision algorithm for *any* fixed g and d [16].

Similarly, the family of "yes" instances for the *k-vertex integrity* problem [5, 10] is minor closed and excludes a sufficiently long path, guaranteeing decidability in $O(n^2)$ time. Finally, observe that the *k-longest path* problem [22] is amenable to this approach [8], this time with the family of "no" instances minor closed and excluding the path of length k (its only obstruction), ensuring decidability in $O(n^2)$ time.

4. *Consequences and Even Better Tools*

Thus, we may now bury the "folk wisdom" that whatever is not obviously easy must be NP-hard. We believe that the most exciting aspect of Theorems A and B is the way in which they constitute a new paradigm for determining the complexity of difficult combinatorial decision problems. The general form of these results is as follows. Let $(S, \leq)$ be a partially ordered set.

THEOREM A* (Polynomial-time order test). *For every fixed $y \in S$, the problem that takes as input $x \in S$ and determines whether $y \leq x$ is solvable in polynomial time.*

THEOREM B* (Well-partial ordering). *Any subset F of S has a finite number of minimal elements.*

When such theorems exist, and when F is closed under the partial order (that is, x in F and $y \leq x$ together imply that y is in F), it follows that F possesses a polynomial-time decision algorithm. We call a partially ordered set for which Theorems A* and B* hold a *Robertson–Seymour poset.* In addition to graphs under the minor ordering, what other posets are Robertson–Seymour?

One example from algebra concerns "valuated" trees. (These are a restricted form of vertex-labeled tree. See, e.g., [39] for a precise definition.) A partial ordering of these combinatorial objects by a relation used in the classification of subgroups of finite Abelian groups [34] has been shown in [26] to be a well-partial order and in [1] to have a polynomial-time order test for every fixed valuated tree.

It has recently been announced (P. D. Seymour, personal communication) that Theorems A* and B* hold for graphs under the *immersion* order, in which $H \leq G$ if and only if there is an injection f: $V(H) \rightarrow V(G)$ and a set of mutually edge-disjoint paths in G joining the images of vertices adjacent in H. This settles in the affirmative a conjecture by Nash-Williams [33]. Binary matroids under the matroid-minor order [46] are conjectured (P. D. Seymour, personal communication) to be a Robertson–Seymour poset as well.

Another possibility is series-parallel graphs under the *induced-minor ordering,* in which $H \leq G$ if a graph isomorphic to H can be obtained from a vertex-induced subgraph of G by contracting edges. Although it is known that in general there exist families of graphs having an infinite number of induced-minor minimal elements, it has been shown in [43] that series-parallel graphs are well-partially ordered by this relation. Whether this poset has a polynomial-time order test for every fixed H is an open question.

There are a number of ways to construct well-partial orders. Applying these constructions to Robertson–Seymour posets can yield additional Robertson–Seymour posets.

THEOREM 7. *Suppose* $(S_1, \leq_1)$ *and* $(S_2, \leq_2)$ *are Robertson–Seymour posets. Then so are the following*:

(*i*) $(S_1 \times S_2, \leq)$ *with* $(a, b) \leq (a', b')$ *if and only if* $a \leq_1 a'$ *and* $b \leq_2 b'$,
(*ii*) *finite sequences of elements of* S_1 *with* $(a_1, a_2, \ldots, a_k) \leq (b_1, b_2, \ldots, b_n)$ *if and only if there exists a strictly increasing function* f: $\{1, 2, \ldots, k\} \to \{1, 2, \ldots, n\}$ *such that* $a_i \leq_1 b_{f(i)}$ *for* $1 \leq i \leq k$,
(*iii*) *finite subsets of* S_1 *with* $F \leq G$ *if and only if there exists an injection* h: $F \to G$ *such that* $a \leq_1 h(a)$ *for all* a *in* F.

PROOF. It is known that (i), (ii), and (iii) describe well-partial orders [32], so that only polynomial-time order tests need to be established.

(i) To determine whether $(a, b) \leq (a', b')$ for fixed (a, b), we simply apply the two polynomial-time tests to determine whether $a \leq_1 a'$ and $b \leq_2 b'$.
(ii) For fixed $(a_1, \ldots, a_k)$, to determine whether $(a_1, \ldots, a_k) \leq (b_1, \ldots, b_n)$, we perform k tests of whether $a_i \leq_1 b_{f(i)}$ for each strictly increasing function f: $\{1, \ldots, k\} \to \{1, \ldots, n\}$. Since the element $(a_1, \ldots, a_k)$ is fixed, there are fewer than n^k collections, each of k polynomial-time tests, to perform.
(iii) Analogous to (ii). □

One method of employing the knowledge that a poset $(S, \leq)$ is Robertson–Seymour in a proof of polynomial-time complexity is to argue directly, as we did for problems in the preceding section, that the family $F \subseteq S$ of inputs for which the answer to the decision problem is "yes" (or the family S–F for which the answer is "no") is closed under the order $\leq$ in S.

The following consequences offer less direct methods.

CONSEQUENCE 1. *If* $(S, \leq)$ *is Robertson–Seymour and if* T *is a subset of* S *such that*

(*a*) $F \subseteq T$ *and*
(*b*) x *in* F, $y \leq x$ *and* y *in* T *together imply* y *must be in* F,

then there is a polynomial-time decision algorithm to determine for input x *in* T *whether* x *is in* F (*for fixed* F).

PROOF. Let M denote the (finite) set of minimal elements in the complement of F in T. Hypothesis (b) implies that, if x is in T, then x is in F if and only if it is not the case that $z \leq x$ for some z in M. Since $(S, \leq)$ is Robertson–Seymour and M is finite, this can be determined in polynomial time. □

CONSEQUENCE 2. *If* $(S, \leq)$ *is Robertson–Seymour and if there is a polynomial-time computable map* t: $D \to S$ *such that for* $F \subseteq D$

(a) $t(F) \subseteq S$ *is closed under* $\leq$ *and*
(b) *either* $t^{-1}(x) \cap F = \varnothing$ *or* $t^{-1}(x) \subseteq F$ *for each* x *in* $t(D)$,

then there is a polynomial-time decision algorithm to determine for input z *in* D *whether* z *is in* F (*for fixed* F).

PROOF. First compute $t(z)$. There is a finite set M of minimal elements in the complement of the image of F, because S is well-partially ordered. By hypothesis (a), $t(z)$ is in the image of the family F if and only if it is not the case that $y \leq t(z)$ for some Y in M. This can be determined in polynomial time by performing an order test for each element of M. By hypothesis (b), z belongs to F if and only if $t(z)$ belongs to the image of F. □

Consider now the following question of near-planarity.

k-crossing number

Instance: A graph G.

Question: Can G be drawn in the plane so that at most k pairs of edges cross one another?

THEOREM 8. *k-crossing number can be decided in $O(n^3)$ time for input restricted to graphs of maximum degree* 3.

PROOF. We employ Consequence 1, with T, the set of graphs of maximum degree 3, and F, the family of "yes" instances.

For graphs of maximum degree 3, the minor ordering of graphs coincides with the *topological* ordering. A graph G contains a graph H topologically if G has a subgraph isomorphic to a graph obtained from H by replacing edges with paths (called *subdivisions* of H). If a subgraph of G can be contracted to H, where H has maximum degree 3, then for each edge uv contracted, either u, v, or both must have degree at most 2. Such a contraction may be viewed as *removing* a subdivision. Thus, for an arbitrary graph G, $G \geq H$ in the minor ordering, where H has maximum degree 3, if and only if $G \geq H$ in the topological ordering.

Any subgraph of a graph that can be drawn with no more than k crossings can also be drawn with no more than k crossings. It remains only to check that the family of graphs that can be drawn with no more than k crossings is closed under subdivision removal. But this is immediate, since if G', a subdivision of G, can be so drawn, then G can as well. □

From the above argument, we see that for input restricted to graphs of maximum degree 3, it suffices to check that the family of "yes" instances is closed under taking subgraphs and removing subdivisions. This fact can be useful for other problems.

The following problem has been shown to be decidable in time $O(n^k)$ for arbitrary graphs by dynamic programming [29].

k-topological bandwidth

Instance: A graph G.

Question: Is there a subdivision H of G that admits a 1 : 1 labeling f: $V(H) \rightarrow \{1, \ldots, |V(H)|\}$ with $\max\{|f(x) - f(y)| : xy \in E(H)\} \leq k$?

THEOREM 9. *k-topological bandwidth can be decided in $O(n^2)$ time for input restricted to graphs of maximum degree* 3.

PROOF. The family of "yes" instances is plainly closed under taking subgraphs. It is also trivially closed under removing subdivisions. It remains only to note that for each value of k there are planar graphs (e.g., $K_{1,2k+1}$) with topological bandwidth greater than k. □

Recall the *g-unit cycle avoidance* problem defined in Section 3.

THEOREM 10. *For an arbitrary Abelian group g, g-unit cycle avoidance can be decided in $O(n^3)$ time for input restricted to graphs of maximum degree* 3.

PROOF. We again use Consequence 1 and the fact that for graphs of maximum degree 3, $G \geq H$ if and only if G has a subgraph isomorphic to a subdivision of H.

Clearly, if G has a labeling that avoids unit cycles, then the same labeling when restricted to any subgraph of G also avoids unit cycles. If G is obtained from some graph G' by subdividing a single edge uv of G' with a vertex s, and if l is a labeling of G that avoids unit cycles, then the labeling l' of G' with $l'(uv) = l(us) + l(sv)$ that otherwise coincides with l avoids unit cycles in G'. □

As an application of Consequence 2, we refer the reader to [16] in which our results can be interpreted as using D to denote a class of Boolean matrices and F to represent the family of "yes" instances of D. There we employed earlier nonconstructive tools [35] to prove that an important fixed-width combinatorial problem [9, 12] of VLSI design known variously as *gate matrix layout*, *multiple PLA folding*, *Weinberger array layout*, and *one-dimensional logic array layout* is in P. We observe that Lemma 3 of [16] guarantees that this decision problem can in fact be solved in quadratic time.

5. *Discussion and Directions for Future Research*

These new tools employing Theorems A and B are nonconstructive in two ways. First, a guarantee of polynomial-time decidability does not produce the decision algorithm. That is, although we are assured that some polynomial-time algorithm always decides correctly, we do not know which algorithm is the right one. (We observe that this situation is not new, for a trivial reason: Celebrated problems such as *Fermat's conjecture* [25, p. 179] are decidable in constant time by one of two algorithms, but we do not know which of the two algorithms answers it correctly.) Are there constructive analogs of these tools useful in building fast decision algorithms?

Second, even if we are able to find the correct polynomial-time decision algorithm, there is no assurance that it will be of any use in solving an optimization version of the problem at hand. *Self-reducibility*, the process by which a decision algorithm may be used to devise a construction algorithm, has until now been primarily a theoretical curiosity (see, e.g., [31]). In almost all previously known cases, polynomial-time decision algorithms proceed by attempting the construction itself. Developments such as those described in this paper suggest that the issue of self-reducibility may well be of practical significance [8, 18].

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